MATH 2060B: Mathematical Analysis II **Mid-Term Test**

Answer ALL Questions
20 points for each question 05 March, 2017. 8:30-10:00

1. Let $f : (a, b) \to \mathbb{R}$ be a C^2-function defined on (a, b).
   (i) Show that if the second derivative $f^{(2)}(x) > 0$ for all $x \in (a, b)$, then f is a strictly convex function on (a, b), that is, $tf(x) + (1 - t)f(y) \geq f(tx + (1 - t)y)$ for all $x, y \in (a, b)$ with $x \neq y$ and $t \in (0, 1)$.
   (ii) Show that $x \log x + y \log y \geq (x + y) \log \frac{x+y}{2}$ for all $x > 0$ and $y > 0$ with $x \neq y$.
   (iii) Show that $\frac{1}{2}(x^p + y^p) \leq \left(\frac{x+y}{2}\right)^p$ for $x > 0, y > 0$ and $0 < p < 1$.

2. (i) Let f be a differentiable function defined on a bounded interval (a, b). Show that if f is unbounded, then so is its derivative f'.
   (ii) Does the converse of (i) hold?
   (iii) If the assumption of boundedness of (a, b) is removed in (i), does (i) still hold?

3. Fix a sequence (x_n) in $[0, 1]$. For each $n = 1, 2...$, let $f : [0, 1] \to [0, 1]$ be a function defined by $f_n(x) = 1$ for $x \in [0, 1] \setminus \{x_1, ..., x_n\}$; otherwise, $f_n(x) = 0$.
   (i) Show that f_n is a Riemann integrable function over $[0, 1]$.
   (ii) Find $\int_0^1 f_n(x)dx$.
   (iii) Suppose that $f(x) = \lim_n f_n(x)$ exists for all $x \in [0, 1]$. Is the function f integrable over $[0, 1]$?

End